# 720: INTRODUCTION TO TOPOLOGICAL PHASES IN CONDENSED MATTER

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**Syllabus**

GENERAL INTRODUCTION

Bibliography and further reading. *For cute ideas about topological insulators, J. Moore, "The birth of topological insulators"* - https://www.nature.com/articles/nature08916. *For a birds-eye view on topological matter in general, X. G. Wen, "Choreographed entangle dances: topological states of quantum matter"* https://arxiv.org/pdf/1906.05983.pdf. *For the advanced reader, Stephan Rachel, "Interacting topological insulators: a review"*, https://arxiv.org/abs/1804.10656.

## TOPOLOGICAL INSULATORS

CHAPTER 1: BERRY PHASE

- 1: Discrete case
  - a: Simple results
  - b: Berry flux
  - c: Chern number
- 2: Continuous case
  - a: Berry phase
  - b: Berry curvature
  - c: Chern number
- 3: Berry phase and adiabatic dynamics
- 4: Formulas for Berry curvature
- 5: Two-level systems
- 6: Aharonov-Bohm effect

Bibliography and further reading. *The equivalent of Berry phase in classical electrodynamics: Rod Cross 2022 Phys. Educ. 57 013003. An article by M. Berry* - https://michaelberryphysics.files.wordpress.com/2013/07/berry187.pdf. *A pedagogical review on Berry phase with experimental applications* - https://www-old.mpi-halle.mpg.de/mpi/publi/pdf/6325_05.pdf.

*Feynman's discussion of the Aharonov-Bohm effect* - https://www.feynmanlectures.caltech.edu/II_15.html. *For a derivation and discussion of the adiabatic theorem, see e.g.* https://www.physics.udel.edu/~jim/PHYS425_20S/Class%20Notes/Notes_19.pdf. (*Note that this is important in the somewhat unrelated topic of adiabatic quantum computing* https://arxiv.org/abs/1611.04471). *For an inspiring article connecting Foucault pendulum to the Berry phase and more see:* https://arxiv.org/pdf/2006.08488.pdf.

Homework 1: Berry phase for two-state systems

CHAPTER 2: SOME SOLID STATE PHYSICS REMINDERS

- 1: Lattices
  - a: Bravais lattices; reciprocal lattice
  - b: Primitive cells, Brillouin zone
- 2: Bloch theorem
- 3: Wave function normalization
- 4: Two examples: the SSH and the Kane-Mele model

Bibliography and further reading. *The book by Ashcroft and Mermin contains everything and more* - https://en.wikipedia.org/wiki/Ashcroft_and_Mermin. *The SSH model results naturally from the existence of the Peierls instability. Look up:* https://en.wikipedia.org/wiki/Peierls_transition.

Homework 2: Second quantization. Wannier wave functions.

CHAPTER 3: A FIRST LOOK AT SYMMETRIES

- 1: Symmetries in general
  - a: The unitary case
  - b: Anti-linear operator
  - c: Anti-unitary symmetries
- 2: Time reversal symmetry
  - a: Generalities
  - b: TRS with spin
- 3: TRS in second quantization
- 4: TRS in momentum space
  - a: Generalities
  - b: TRS and Berry phase
- 5: Space (inversion) symmetry
- 6: Chiral (sublattice) symmetry

Bibliography and further reading. *Some mathematical properties of anti-linear operators* - https://aip.scitation.org/doi/10.1063/1.1705348.

Homework 3: Time-reversal symmetry in various (quantum) settings

CHAPTER 4: PHENOMENOLOGY OF TOPOLOGICAL PHASES FROM 1D EXAMPLES

- 1: Topological properties of the SSH model
- 2: Edge states
- 3: Edge states from the Dirac equation
- 4: Polarization, Berry and Zak phase
- 5: The invariant from inversion eigenvalues
- 6: Wannier centers and charge pumping
- 7: 1D insulators as obstructed insulators
- 8: The Kitaev wire

Bibliography and further reading. *Experiments with the SSH model* - https://www.nature.com/articles/ncomms13986 and https://arxiv.org/abs/1502.00059. *A review by Jackiw about charge fractionalization* - http://physics.gu.se/~tfkhj/Jackiw%20on%20SSH.pdf. *A short book about the SSH model* - https://physics.bme.hu/sites/physics.bme.hu/files/users/BMETE11MF34_kov/SSH-2021Fall.pdf. *A review about Majorana fermions* -https://arxiv.org/pdf/1202.1293.pdf. *Studying the spectrum of finite Kitaev chains is painful but very instructive. See for instance* https://arxiv.org/abs/1909.10971.

Homework 4: Edge states in the SSH model.

CHAPTER 5: A SECOND LOOK AT SYMMETRIES

- 1: Charge conjugation symmetry
- 2: Chiral symmetry revisited
- 3: The ten-fold way
- 4: Examples
    - a: The SSH model and the ten-fold way

Bibliography and further reading. *Articles about the ten-fold way: A.W.W. Ludwig, " Topological phases: Classification of topological insulators and superconductors of non-interacting Fermions, and beyond"* - https://arxiv.org/abs/1512.08882. *For a thorough discussion of which class the SSH model belongs to - and what it means - see:* https://physics.stackexchange.com/questions/411787/mathbb-z-2-or-mathbb-z-invariant-for-the-su-schrieffer-heeger-ssh-model. *The talk by Ludwig* - https://online.kitp.ucsb.edu/online/freedmanfest/ludwig/pdf/Andreas_Ludwig.pdf *contains many interesting remarks in particular about the interpretation of the classification.*

CHAPTER 6: CHERN INSULATORS IN 2D

- 1: Chern insulators and the Chern number
  - a: Generalities
  - b: Example of a Chern insulator: the "spin-less BHZ model"=SBHZ
- 2: Chern number and the Hall conductivity
- 3: Bulk-boundary correspondence. Chiral boundary modes
- 4: Quantum Hall state as a topological insulator
  - a: Graphene: basics
  - b: The Haldane model
  - c: Chern number for the Haldane model

Bibliography and further reading. *We mostly saw values $C = 0, \pm 1$ in class, but it is possible to reach higher Chern numbers by adding next to next neighbor interactions. This is discussed for instance in the paper by Sticlet et al., "Geometrical engineering of a two-bands Chern insulator in two dimensions with arbitrary topological index":* https://arxiv.org/abs/1201.6613. *Physical realizations of the toy SBHZ model (and others) are discussed in the paper by Bernegiv et al., "Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells":* https://arxiv.org/pdf/cond-mat/0611399.pdf. *If you wish to learn more about the quantum Hall effect, I recommend T. Dong's lectures:* https://www.damtp.cam.ac.uk/user/tong/qhe.html. *A more elementary summary (with historical bits) is provided by the following article in Physics Today:*https://physicstoday.scitation.org/doi/10.1063/1.1611351.

Homework 5: Topological invariants

CHAPTER 7: TIME REVERSAL SYMMETRIC INSULATORS IN 2D

- 1: The BHZ model and quantum spin Hall
- 2: The Kane Mele model

Bibliography and further reading. *The issue of the relationship between non-trivial topology and Wannier wave functions is reviewed in the nice paper "Wannier representation of Z2 topological insulators"* https://www.physics.rutgers.edu/~dhv/pubs/local_copy/as_z2.pdf

CHAPTER 8: 3D TOPOLOGICAL INSULATORS

Bibliography and further reading. *A very good early article describing 2D and 3D topological insulators, invariants, Dirac cones etc:* https://arxiv.org/abs/0710.0930v1. *Strong and weak topological insulators:* https://arxiv.org/abs/1105.4351.

CHAPTER 9: WEYL SEMIMETALS

I decided not to discuss the topic and move on to slightly different things for now.

Bibliography and further reading. *But if you are curious look up the review by S. Rao:* https://arxiv.org/abs/1603.02821

## CHAPTER 10: TOPOLOGICAL MATTER IN 1D

- 1: The AKLT spin chain - ground state
- 2: Matrix Product states
- 3: The AKLT spin chain - physical properties
- 4: SPT phases

Bibliography and further reading. *The original AKLT paper - while not aware of the depth of its results - is beautifully simple, and worth a look:* https://link.springer.com/content/pdf/10.1007/978-3-662-06390-3_19.pdf . *For a general discussion of the AKLT chain and the Haldane phase, the slides by M. Oshikawa are very useful:* https://online.kitp.ucsb.edu/online/compqcm10/oshikawa/pdf/Oshikawa_CompQCM.pdf. *For the phase diagram of the bilinear, biquadratic chain, look up the papers "Period tripling in the bilinear-biquadratic antiferromagnetic S=1 chain" and "A New Phase of the S=1 Bilinear-Biquadratic Chain: -a Trimerized State":* https://doi.org/10.1103/PhysRevB.44.11836 *and* https://journals.jps.jp/doi/10.1143/JPSJ.60.389.

## CHAPTER 11 (what would be) AND BEYOND

For a general discussion of topological matter, including a useful historical perspective, look at the review "Zoo of quantum topological phases of matter" by X.G. Wen: https://arxiv.org/abs/1610.03911.

## BIBLIOGRAPHY

On top of the research and review articles mentioned in each chapter, here are a few general suggestions:

- B. A. Bernevig and T. Hughes, "Topological insulators and topological superconductors", Princeton University Press
- R. Shankar, "Topological insulators, a review", https://arxiv.org/abs/1804.06471
- R. Grushin, "Introduction to topological matter", https://grushingroup.cnrs.fr/topointro2021/