QUANTUM SPIN CHAINS

Technical Syllabus[1]

Two notes:

- The syllabus with proper administrative details is available on BlackBoard
- There will be some more material posted on blackboard, together with copies of some of the references mentioned here.

Chapter 1: Spin wave theory for the XXX spin chain

- Large s analysis: bosonic representation of spin operators
- Spin waves in the ferromagnetic case
- Spin waves in the antiferromagnetic case
- Coleman's theorem. Some rigorous results

Suggested reading: For *bosonic representations*, have a look at the Wikipedia article on Jordan (or Jordan-Schwinger) maps. For some *rigorous results on Neel ground states*, the article by Kennedy et al. (J. Stat. Phys. 53 (1988) 1019) is a good starting point. For some *numerics* in $d = 2$ and $s = \frac{1}{2}$, see the article by Sandvik (arXiv:cond-mat/9707123). For a good review on *spin waves* (including phenomenological aspects we are not really interested in here) see the article by Kranendonk and Van Vleck, Rev. Mod. Phys. 30 (1958) 1 (I made it available on blackboard). Finally, for the Coleman-Mermin-Wagner theorem, you can look up the wikipedia article (https://en.wikipedia.org/wiki/MerminWagner_theorem) , or the beautiful paper by S. Coleman, Comm. Math. Phys. 31 (1973) 259.

Chapter 2: Lieb-Schultz-Mattis theorem.

- Ground state and symmetries
- Building low-lying excited states
- The "theorem"

Suggested reading: we will discuss more Hamiltonians (such as the Majumdar Ghosh model) whose ground states can easily be found using $SU(2)$ *representation theory*. In the meantime, you can look for instance at the paper by Michaud et al. (Phys. Rev. Lett. 108, 127202 (2012)). For an *extension of the Lieb Schulz Mattis theorem* in higher dimensions, see the paper by Hastings (arxiv.org/pdf/cond-mat/0305505.pdf). Another direction of potential interest to some of you may be the *Lieb-Robinson bounds*: see the wikipedia article (en.wikipedia.org/wiki/Lieb-Robinson_bounds) as a starting point.

Intermezzo I: quantum versus classical - Hamiltonians versus transfer matrices

- Classical Ising model. Transfer matrix.
- The Hamiltonian limit.

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Suggested reading: section III of the review article by Kogut (Rev. Mod. Phys. 51, 659 (1979)) is superb. You can also look at the article by Fuchs where things are worked out in great detail for the Ising model in d-dimensions (N.H. Fuchs, Phys. Rev. B 35, 2045 (1987))

Chapter 3: Large s limit and mapping onto sigma model

- Quantum spins and coherent states
- The path integral for a single spin. The Wess-Zumino term
- The path integral for many spins
- The non-linear sigma model and the Θ term

Suggested reading: for spherical trigonometry, you'll find plenty of formulas in the Wikipedia and Wolfram articles on the topic. For details how to prove these formulas, see for instance the notes by Rob Johnson, https://www.math.ucla.edu/ robjohn/math/spheretrig.pdf. For the Riemann sphere, look at the wikipedia article https://en.wikipedia.org/wiki/Riemann_sphere. For basic results about the homotopy groups of sphere, look again at wikipedia https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres. For a review on the difference between integer and half-integer spin Heisenberg chains, look up the excellent review article by I. Affleck, J. Phys. C1 (1989) 3047. The early article about instantons by Belavin and Polyakov (JETP Letters 22 (1975) 246) is also very inspiring.

Chapter 4: The case s=1/2: the Bethe-ansatz solution.

- Reformulations of the chain. Symmetries and permutations.
- One spin down
- Two spins down
- The general solution
- The physics of the solution

Suggested reading: for a discussion of magnons and spinons, see https://physics.stackexchange.com/questions /174341/what-is-the-difference-between-a-magnon-and-a-spinon. For a pedestrian review of the Bethe-ansatz in the XXX chain (with a somewhat more physical approach) see the three papers by Karbach and Müller, arXiv:cond-mat/9008162, 9809163 and 0008018. For those of you brave enough to learn about the algebraic Bethe-ansatz, the review by Fadeev is the classic reference. It can be found for instance at https://cds.cern.ch/record/261263/files/9404013.pdf. A lucid discussion of the issus of completeness can be found in the paper by Baxter arXiv:cond-mat/0111188.

Chapter 5: A very short discussion of RG flows and the Haldane conjecture

Suggested reading: on top of Affleck and Haldane papers, I recommend the very lucid work by Shankar and Read, Nucl. Phys. B336 (1990) 457.

Intermezzo 2: the Jordan Wigner transformation.

Chapter 6: The AKLT model

- The ground state

- First properties of spin configurations
- Normalization of the ground state
- Correlation functions
- The ground state as an MPS state

Suggested reading: the Bethe-ansatz states turn out to be in fact MPS states. This is a little hard to see without the formalism of algebraic Bethe-ansatz, but it's possible. See the paper by Alcaraz and Lazo, arXiv:cond-mat/0304170. Apart from the seminal AKLT work (Affleck et al., Comm. Math. Phys. 115 (1988) 477), there are useful calculations in a paper by Den Nijs and Rommelse, Phys. Rev. B40 (1989) 4709. Many rigorous aspects are further explored in a paper by Kennedy and Tasaki in Comm. Math. Phys. 147 (1992) 431.

Chapter 7: Properties of the Haldane phase

- Symmetries and order parameter in the AKLT model

  Edge states in the Haldane phase
- The phase diagram of the spin 1 chain

Suggested reading: entanglement in the AKLT model is discussed for instance in the paper 1806.09624. The $Z_2 \times Z_2$ symmetry is discussed at length in the paper by Kennedy Tasaki mentioned in the previous chapter. For numerical studies of the Haldane phase ,see e.g. the papers by Botet et al. Phys. Rev. B28 (1983) 3914, and the one by Hatsugai anf Kohmoto, Phys. Rev. B44 (1991) 11789. Interesting more general aspects (in particular for higher spin) appear in the paper by Oshikawa, J. Phys. C4 (1992) 7469.

Chapter 8: Topological phases

- Relation with the SSH model
- "Winding numbers" and edge states
- Symmetries
- The concept of topological phases

Suggested reading: The SSH model is discussed in many places. The review by Batra and Sheer arXiv:1906.08435 is a useful example. For a more fancy but very deep point of view, see the slides of the talk by Wen on blackboard. Wen has also written a more general and less technical review, also available on blackboard. The discussion of MPS states from an RG point of view in the paper by Verstraete et al. quant-ph/0410227 is particularly inspiring.

Chapter 9: Random chains

- The real space RG
- Flow of the probability distribution
- Entanglement entropy

Suggested reading: the whole lecture was based on the excellent paper by Refael and Moore, arXiv:cond-mat/0406737: it's really self-contained, and covers most of the interesting aspects.

Chapter 10: Bosonization

- The field theory limit of the XX chain in fermionic variables.
- Commutation relations for the density modes - a first look at bosonization.
- The free boson.
- Normal ordering for free fermions and free bosons
- The bosonization rules
- Ljuttinger liquids

Suggested reading: the collection of reprints "The many-body problem" by D. C. Mattis contains many relevant references in chapter 4 in particular. The classical paper is Mattis and Lieb J. Math. Phys. 6 (1965) 304 discusses what went wrong in Luttinger's solution, which did not pay enough attention to the role of the Fermi sea. There are many references on bosonization; for a more physical approach complementing ours, see T. Giamarchi, 'Quantum physics in one dimension' (Oxford). For classic references on the field theoretic side, see S. Coleman, Phys. Rev. D 11 (1975) 2088 and S. Mandelstam, Phys. Rev. D11 (1975) 3026.